FLOW STABILITY OF A SECOND-ORDER STOKES FLUID IN A PLANE CHANNEL

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The stability of a second-order Stokes fluid has only recently begun to attract the attention of researchers; it is therefore understandable that we have not encountered even one study in this field. In beginning our study of this problem we shall use the classification proposed in [1]; for this classification we should distinguish between a second-order Stokes fluid and a generalized Reiner-Rivlin fluid since the latter possesses a "memory." For the most part, studies concerned with the flow stability of such a fluid center around flow along an inclined plane [2,3].

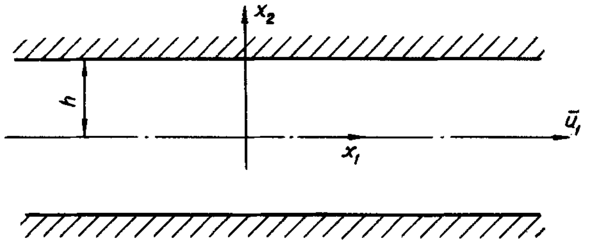

Fig. 1. Illustrative diagram.

In this article we shall consider the flow stability of weakly linear fluids with respect to long-wave perturbations in a plane channel.

Figure 1 shows the flow pattern. Without allowing for the external volume forces we write the equation of motion as

$$\rho\left(\frac{\partial u_i}{\partial t}+u_a\frac{\partial u_i}{\partial x_a}\right)=\frac{\partial \tau_{ia}}{\partial x_a}. \tag{1}$$

We assume the flow is incompressible, isotropic, and satisfies the equation of state

$$\tau_{ia}=f_0(I)\,\delta_{ia}+f_1(I)e_{ia}+f_2(I)\,e_{ik}e_{ka}, \tag{2}$$

where

$$e_{ia}=\frac{\partial u_i}{\partial x_a}+\frac{\partial u_a}{\partial x_i};$$

I is the common symbol denoting the three invariants of the tensor e_{ia}; f_0, f_1, and f_2 are functions of the three invariants specified as

$$f_0(I)=-p,\quad f_1(I)=\mu-\mu_2 I_2,\quad f_2(I)=\mu_3. \tag{3}$$

In these expressions, p is the pressure and μ, μ_2, and μ_3 are the physical constants of the fluid.

Assuming that the unperturbed flow has only one velocity component $\bar{u}_1$ directed along the x-axis (Fig. 1) we therefore have $I_3 = 0$ and $I_2 = -(d\bar{u}_1/dy)^2$. Then, after some transformations using the method of successive approximations we obtain the velocity profile for unperturbed motion

$$U=\frac{1}{2}(1-y^2)-\frac{1}{4}M(1-y^4)+\frac{1}{2}M^2(1-y^6), \tag{4}$$

where $U = \mu\bar{u}_1/kh^2$; $k = -p$; $M = (\mu_2/\mu_3)(kh)^2$ is a parameter for fluid nonlinearity; below, we shall assume this parameter has intermediate values.

Using the usual method [2] we write the Orr-Sommerfeld equation and the boundary conditions for this problem as

$$\Phi^{IV}-2\alpha^2\Phi''+\alpha^4\Phi+3M(U'^2\Phi'')''+2M\alpha^2(U'^2\Phi''+$$
$$+2U'U''\Phi'+3U''^2\Phi+3U'U'''\Phi)+3M\alpha^4U'^2\Phi=$$
$$=i\alpha R[U(\Phi''-\alpha^2\Phi)-U''\Phi]-i\beta R(\Phi''-\alpha^2\Phi), \tag{5}$$

$$\Phi(\pm 1)=\Phi'(\pm 1)=0. \tag{6}$$

The characteristic values for this problem are specified by Eq. (5) and boundary conditions (6). For a nontrivial solution, we must have

$$\beta=\beta(R,\ M,\ \alpha).$$

The equation

$$\beta_i(R,\ M,\ \alpha)=0 \tag{7}$$

determines the relationship between R and α for a given value of M. Graphically, this relationship is the neutral-stability curve.

We shall solve this problem by the method of successive approximations.

Assuming α and M are on the order of unity, in the zeroth approximation we have

$$\Phi_0^{IV}+i\beta_0 R\,\Phi_0''=0, \tag{8}$$

whose integral is

$$\Phi_0=A\,\mathrm{sh}py+B\,\mathrm{ch}py+Cy+D, \tag{9}$$

where

$$p^2=-i\beta_0 R.$$

Allowing only for the even part of Φ [4] and using the boundary conditions to find the constant we can write the characteristic functions and characteristic values in the zeroth approximation with an accuracy to within an arbitrary constant:

$$\Phi_0=\mathrm{ch}\,py-\mathrm{ch}\,p, \tag{10}$$

$$\beta_{0i}=-i\,\frac{(m\pi)^2}{R},\quad m=1,\ 2,\ 3,\ \ldots. \tag{11}$$

From the last equation it follows that in the zeroth approximation motion is stable for any R (since $\beta_{i0} < 0$).

Performing similar calculations for the first approximation and restricting ourselves to the second approximation we obtain characteristic functions and characteristic values in the first and second approximations. Here we shall only write the increments in the characteristic values obtained respectively in the first and second approximations:

$$\beta_1=\beta_{1r}+i\beta_{1i}=\alpha\left[\frac{1}{3}-\frac{5}{4}\,\frac{1}{(m\pi)^2}\right]-$$
$$-i\,\frac{M}{R}\left[\frac{3}{2}+(m\pi)^2\right], \tag{12}$$

$$\beta_{2i}=\frac{\alpha^2 R}{(m\pi)^2}\left[\frac{1}{180}+\frac{7}{48}\,\frac{1}{(m\pi)^2}-\frac{77}{32}\,\frac{1}{(m\pi)^4}\right]. \tag{13}$$

Adding the imaginary parts of the characteristic values of all approximations (11), (12), and (13) we obtain the following expression for the growth coefficient:

$$\beta_i=\alpha^2\,\frac{R}{(m\pi)^2}\left[\frac{1}{180}+\frac{7}{48(m\pi)^2}-\frac{77}{32(m\pi)^4}\right]-$$
$$-\frac{1}{R}\left[(m\pi)^2-\alpha^2\right]-\frac{M}{R}\left\{\frac{3}{2}+(m\pi)^2+\right.$$
$$\left.+M\left[\frac{9}{5}(m\pi)^2+\frac{27}{4}-\frac{81}{8(m\pi)^2}\right]\right\}. \tag{14}$$